ON UNIFORM CONVERGENCE AND A SERIES OF VARIABLE TERMS

A THESIS

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PREFACE

This thesis is a brief study on the theory of uniform convergence. It contains material which ultimately leads to the analysis of a series of variable terms.

Chapter I contains basic information relative to uniform convergence needed in the remainder of the thesis. This information includes definitions and examples on uniform convergence.

Chapter II contains the analysis of a series of variable terms. The discussion of this series is the main objective of this thesis.

The information included in Chapter III gives a brief summary of the thesis, and it discusses the series considered in Chapter II. Furthermore, it discusses the series' connection with the theory of uniform convergence; that is also included in Chapter II.

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GLOSSARY OF SYMBOLS AND ABBREVIATIONS

$\in$ "belongs to"

$\rightrightarrows$ "approaches"

$\Longrightarrow$ "implies"

$\longrightarrow$ "tends to"

Q.E.D. (quod erat demonstrandum-"which was to be proved")

$\sum$ "sigma," summation

CHAPTER I

PRELIMINARY REMARKS ABOUT UNIFORM CONVERGENCE

<u>Uniform convergence of sequences.</u>

Let $$S_1(x),\ S_2(x),\ldots,S_n(x),\ldots$$
be a sequence of functions defined on a set A. Each term of the sequence is a function of x and we denote it by $S_n(x)$. Thus $S_n(x)$ will converge on A if for every $x \in A$ the sequence of constants $\{S_n(x)\}$ converges. So then we assume $S_n(x)$ converges on A and define

$$S(x) \equiv \lim_{n \to +\infty} S_n(x).$$

Then the rapidity with which $S_n(x)$ approaches $S(x)$ can be expected to depend on the value of x.[1]

<u>Definition</u>.-A sequence of functions $\{S_n(x)\}$, defined on a set A, converges uniformly on A to a function $S(x)$ defined on A, this being written

$$S_n(x) \rightrightarrows S(x),$$

if and only if corresponding to $\epsilon > 0$ there exists a number $N = N(\epsilon)$, dependent on ϵ alone and not on the point x, such that $n > N$ implies

$$|S_n(x) - S(x)| < \epsilon$$

for all $x \in A$.

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[1] J. M. H. Olmsted, <u>Real Variables</u>, New York, [1959], p. 270.

Example: Suppose $S_n(x) = \dfrac{x}{x + n}$. We want to show that $S_n(x)$ converges uniformly on the interval $[0, b]$. So then for $\epsilon > 0$, we want to show there exists a number $N = N(\epsilon)$, such that

$$|S_n(x) - S(x)| < \epsilon$$

for all $n > N$ and $x \in [0, b]$. Observe that the

$$\lim_{n \to +\infty} S_n(x) = \lim_{n \to +\infty} \frac{x}{x + n} = 0.$$

Then we want $\left|\dfrac{x}{x + n}\right| < \epsilon$ for all $x \in [0, b]$ for $n > N$.

This implies that for $n > N$ $|x| < \epsilon|x + n| \Rightarrow \dfrac{|x|}{\epsilon} < |x + n| \Rightarrow$

$|x + n| > \dfrac{|x|}{\epsilon} \Rightarrow n > \dfrac{x}{\epsilon}(1 - \epsilon)$. If $x = b$, then $n > \dfrac{b}{\epsilon}(1 - \epsilon)$.

Take $N(\epsilon) \equiv \dfrac{b}{\epsilon}(1 - \epsilon)$, and $\left|\dfrac{x}{x + n}\right| < \epsilon$ for all $n > N$ and $x \in [0, b]$.

Q.E.D.

Negation of uniform convergence.—A sequence of functions $\{S_n(x)\}$, defined on a set A, fails to converge uniformly on A to a function S(x) defined on A if and only if there exists a positive number ϵ having the property that for any number N there exist a positive integer $n > N$ and a point $x \in A$ such that

$$|S_n(x) - S(x)| \geq \epsilon.$$

Example: Show that $S_n(x) = \dfrac{x}{x + n}$ does not converge uniformly on $[0, +\infty)$.

Solution: Suppose we take $\epsilon = \frac{1}{2}$. If N is an arbitrary number, let us choose any $n > N$, and hold this n fixed. Next we pick $x > n$. Then for this pair x and n,

$$|S_n(x) - S(x)| = \frac{x}{x+n} > \frac{x}{x+x} = \frac{x}{2x} = \frac{1}{2}.$$

Since $\epsilon \equiv \frac{1}{2}$ this proves that $\frac{x}{x+n}$ does not converge uniformly on $[0, +\infty)$. Q.E.D.

Uniform convergence of series.

Let $$U_1(x) + U_2(x) + \dots + U_n(x) + \dots$$

be a series of functions defined on a set A, and let

$$S_n(x) \equiv U_1(x) + U_2(x) + \dots + U_n(x).$$

We say that this series of functions converges on A in case the sequence $\{S_n(x)\}$ converges on A. The series

$$U_1(x) + U_2(x) + \dots U_n(x) + \dots$$

converges uniformly on A if and only if the sequence $\{S_n(x)\}$ converges uniformly on A.[1]

Definition.-The series $\sum_{n=1}^{\infty} U_n(x)$ is uniformly convergent to S(x) for a set A of values of x if for each $\epsilon > 0$, an integer N can be found such that

$$|S_n(x) - S(x)| < \epsilon \text{ for } n \geq N \text{ and all } x \in A.^{2}$

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[1] J. M. H. Olmsted, Real Variables, New York, [1959], p. 273.

[2] W. Kaplan, Advanced Calculus, Massachusetts, [1952], p. 339.

Thus we see that the uniform convergence of the series $\sum U_n(x)$ is equivalent to uniform convergence of its sequence of partial sums $S_n(x)$.

At this point, we will demonstrate the definition stated above with the use of two complex series $\sum_{n=0}^{\infty} z^n$ when $|z| < 1$ and $\sum_{n=0}^{\infty} \frac{1}{z^n}$ when $|z| > 1$.

This can be done since the interchange of interval for region in dealing with complex numbers does not destroy the essence of the definition.[1]

These two series will be used later in this study to show that a series of variable terms converges to a function of z when $|z| < 1$ and also when $|z| > 1$.

Example 1. Show that the series $\sum_{n=0}^{\infty} z^n$ converges uniformly when $|z| < \rho$ for $\rho < 1$.

Consider $S_n(z) = \sum_{k=0}^{n} z^k = 1 + z + z^2 + z^3 + \ldots + z^n = \frac{1}{1-z} - \frac{z^{n+1}}{1-z}$. Moreover, $\lim_{n\to+\infty} S_n(z) = S(z)$ when $|z| < 1$. Therefore, $\lim_{n\to+\infty} \frac{1}{1-z} - \frac{z^{n+1}}{1-z} = \frac{1}{1-z}$ when $|z| < 1$. By the definition

of uniform convergence, for $\epsilon > 0$, there must exist a positive

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[1] K. Knopp, Theory and Application of Infinite Series, New York, [1928], p. 428.

integer $N = N(\epsilon)$ such that $|S_n(z) - S(z)| < \epsilon$ when $n > N$ and all z such that $|z| \leq \rho$.

Observe $|S_n(z) - S(z)| = \left| \frac{1}{1-z} - \frac{z^{n+1}}{1-z} - \frac{1}{1-z} \right| = \left| \frac{-z^{n+1}}{1-z} \right| = \left| \frac{z^{n+1}}{1-z} \right|$. Take $N = N(\epsilon) \equiv \epsilon(1-\rho)$, and $\left| \frac{z^{n+1}}{1-z} \right| < \epsilon$ for all $n > N$ and all z such that $|z| \leq \rho$. Therefore, the $\sum_{n=0}^{\infty} z^n$ converges uniformly in $|z| \leq \rho$. Q.E.D.

Example 2. Show that the $\sum_{n=0}^{\infty} \frac{1}{z^n}$ converges uniformly in $|z| \geq \rho$ for ρ arbitrary and $\rho > 1$.

Consider $S_n(z) = \sum_{k=0}^{n} \frac{1}{z^k} = 1 + \frac{1}{z} + \frac{1}{z^2} + \ldots + \frac{1}{z^n} = \frac{z}{z-1} - \frac{z-1}{z^n}$. Thus $\lim_{n \to +\infty} \frac{z}{z-1} - \frac{z-1}{z^n} = \frac{z}{z-1}$ for all z such that $|z| > 1$.

We want to show for $\epsilon > 0$, there exists a positive integer $N = N(\epsilon)$ such that for $n > N$ $|S_n(z) - S(z)| < \epsilon$.

Observe $\left| \frac{z}{z-1} - \frac{z-1}{z^n} - \frac{z}{z-1} \right| = \left| -\frac{z-1}{z^n} \right| = \left| \frac{z-1}{z^n} \right|$. If $|z| = \rho$, then take $N(\epsilon) \equiv \epsilon \rho^n$. Now $\left| \frac{z-1}{z^n} \right| < \epsilon$ for all $n > N$ and all z such that $|z| \geq \rho$. Hence the $\sum_{n=0}^{\infty} \frac{1}{z^n}$ converges uniformly in $|z| \geq \rho$. Q.E.D.

CHAPTER II

SERIES OF VARIABLE TERMS

This chapter will deal with a series consisting of variable terms. Thus the value of the series depends on the choice of a definite quantity or variable. We shall consider a series whose terms depend on the variable z, that is, a series of the form $\sum_{n=0}^{\infty} f_n(z)$. This series defines for certain values of z a function f. In particular we consider the series

$\sum_{n=1}^{\infty} \frac{z^{n-1}}{(1-z^n)(1-z^{n+1})}$.[1] We shall show that it is equal to

$\frac{1}{(1-z)^2}$ when $|z| < 1$, and is equal to $\frac{1}{z(1-z)^2}$ when $|z| > 1$.

First we show that the series $\sum_{n=1}^{\infty} \frac{z^{n-1}}{(1-z^n)(1-z^{n+1})} = \frac{1}{(1-z)^2}$ when $|z| < 1$.

Suppose

$$S_n(z) = \sum_{k=1}^{n} \frac{z^{k-1}}{(1-z^k)(1-z^{k+1})}$$

is the sum of the first n terms. So then

[1]E. T. Whittaker, Course of Modern Analysis, Cambridge, 1935, p. 59.

$$(1) \qquad S_n(z) = \sum_{k=1}^{n} \frac{z^{k-1}}{(1-z^k)(1-z^{k+1})} = \frac{1}{(1-z)(1-z^2)} +$$

$$\frac{z}{(1-z^2)(1-z^3)} + \frac{z^2}{(1-z^3)(1-z^4)} + \frac{z^3}{(1-z^4)(1-z^5)} +$$

$$\frac{z^4}{(1-z^5)(1-z^6)} + \dots + \frac{z^{n-1}}{(1-z^n)(1-z^{n+1})} .$$

It follows that

$$(2) \qquad S_n(z) = \frac{1}{(1-z)^2} \left\{ \frac{1}{(1+z)} + \frac{z}{(1+z)(1+z+z^2)} + \right.$$

$$\frac{z^2}{(1+z+z^2)(1+z+z^2+z^3)} + \frac{z^3}{(1+z+z^2+z^3)(1+z+z^2+z^3+z^4)}$$

$$\frac{z^4}{(1+z+z^2+z^3+z^4)(1+z+z^2+z^3+z^4+z^5)} + \dots +$$

$$\left. \frac{z^{n-1}}{\left(\sum_{k=0}^{n-1} z^k\right)\left(\sum_{k=0}^{n} z^k\right)} \right\} .$$

Now it follows from algebra using the theorm on partial fractions, that the fraction $\dfrac{z}{(1+z)(1+z+z^2)} = \dfrac{A}{1+z} + \dfrac{Bz + C}{1+z+z^2}$.

Hence
$$\begin{aligned} z &= A(1+z+z^2) + (Bz + C)\,(1 + z) \\ &= A + Az + Az^2 + Bz + Bz^2 + Cz + C \\ &= (A + B)z^2 + (A + B + C)z + A + C . \end{aligned}$$

Equating coefficients, we have

$$A + B = 0,$$

$$A + B + C = 1,$$

$$A + C = 0.$$

Hence $C = 1$, $A = -1$, and $B = 1$. Therefore,

$$\frac{z}{(1+z)(1+z+z^2)} = -\frac{1}{1+z} + \frac{z+1}{1+z+z^2}.$$

Consider now the fraction

$$\frac{z^2}{(1+z+z^2)(1+z+z^2+z^3)} = \frac{Az + B}{1+z+z^2} + \frac{Cz^2+Dz+E}{1+z+z^2+z^3}.$$

Hence

$$\begin{aligned} z^2 &= (Az+B)\ (1+z+z^2+z^3) + (Cz^2+Dz+E)\ (1+z+z^2) \\ &= Az^4+(A+B)z^3+(A+B)z^2+(A+B)z+B+Cz^4+(C+D)z^3+ \\ &\quad (C+D+E)z^2+(D+E)z+E \\ &= (A+C)z^4+(A+B+C+D)z^3+(A+B+C+D+E)z^2+(A+B+D+E)z+(B+E). \end{aligned}$$

Equating coefficients, we have

$$A + C = 0,$$

$$A + B + C + D = 0,$$

$$A + B + D + E = 0,$$

$$A + B + C + D + E = 1,$$

$$B + E = 0.$$

Hence $A = -1$, $B = -1$, $C = 1$, $D = 1$, and $E = 1$. Therefore,

$$\frac{z^2}{(1+z+z^2)(1+z+z^2+z^3)} = -\frac{z+1}{1+z+z^2} + \frac{z^2+z+1}{1+z+z^2+z^3}.$$

Thus, it is easily seen that

$$\frac{z^{n-1}}{\left(\sum_{k=0}^{n-1} z^k\right)\left(\sum_{k=0}^{n} z^k\right)} = -\frac{\sum_{k=0}^{n-2} z^k}{\sum_{k=0}^{n-1} z^k} + \frac{\sum_{k=0}^{n-1} z^k}{\sum_{k=0}^{n} z^k}.$$

Substituting these results in equation (2), we obtain

$$(3) \qquad S_n(z) = \frac{1}{(1-z)^2}\left\{\frac{1}{1+z} - \frac{1}{1+z} + \frac{z+1}{1+z+z^2} - \frac{z+1}{1+z+z^2} + \frac{z^2+z+1}{1+z+z^2+z^3} + \cdots - \frac{\sum_{k=0}^{n-2} z^k}{\sum_{k=0}^{n-1} z^k} + \frac{\sum_{k=0}^{n-1} z^k}{\sum_{k=0}^{n} z^k}\right\}.$$

Since the first $2n-2$ terms of the sum (3) cancel each other leaving the last term, we have

$$S_n(z) = \frac{1}{(1-z)^2}\left\{\frac{\sum_{k=0}^{n-1} z^k}{\sum_{k=0}^{n} z^k}\right\}.$$

Consider the sums $\sum_{k=0}^{n-1} z^k$ and $\sum_{k=0}^{n} z^k$ for $|z| < 1$.

Both converge to the same sum as $n \to +\infty$, namely $\frac{1}{1-z}$.

Hence, the

$$\lim_{n\to+\infty} \frac{1}{(1-z)^2} \left\{ \frac{\sum_{k=0}^{n-1} z^k}{\sum_{k=0}^{n} z^k} \right\} = \frac{1}{(1-z)^2},$$

when $|z| < 1$, for the expression $\left\{ \frac{\sum_{k=0}^{n-1} z^k}{\sum_{k=0}^{n} z^k} \right\} \to 1$ as $n \to +\infty$,

when $|z| < 1$. Q.E.D.

We will now show that the series

$$\sum_{n=1}^{\infty} \frac{z^{n-1}}{(1-z^n)(1-z^{n+1})} = \frac{1}{z(1-z)^2}$$

when $|z| > 1$. Consider the sum $\sum_{k=0}^{n} z^k$. Thus writing this as $1 + z + z^2 + z^3 + \dots + z^n = \sum_{k=0}^{n} z^k$. Dividing this sum through by z^n and multiplying through by z^n , we have

$$z^n\left(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \dots + \frac{1}{z^n}\right) = \sum_{k=0}^{n} z^k .$$

Thus

$$z^n \sum_{k=0}^{n} \frac{1}{z^k} = z^n\left(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \dots + \frac{1}{z^n}\right)$$

$$= (1 + z + z^2 + \dots + z^n).$$

But the series $\sum_{k=0}^{\infty} \frac{1}{z^k}$ converges for all z when $|z| > 1$.

Now observe

$$S_n(z) = \frac{1}{(1-z)^2}\left\{\frac{z^{n-1}\sum_{k=0}^{n-1}\frac{1}{z^k}}{z^n\sum_{k=0}^{n}\frac{1}{z^k}}\right\} = \frac{1}{z(1-z)^2}\left\{\frac{\sum_{k=0}^{n-1}\frac{1}{z^k}}{\sum_{k=0}^{n}\frac{1}{z^k}}\right\}.$$

Since $\sum_{k=0}^{n-1} \frac{1}{z^k}$ and $\sum_{k=0}^{n} \frac{1}{z^k}$ both converge to the same limit as $n \longrightarrow +\infty$ when $|z| > 1$, namely $\frac{z}{z-1}$.

Therefore, it follows that

$$\lim_{n \to +\infty} \frac{1}{z(1-z)^2} \left\{ \frac{\displaystyle\sum_{k=0}^{n-1} \frac{1}{z^k}}{\displaystyle\sum_{k=0}^{n} \frac{1}{z^k}} \right\} = \frac{1}{z(1-z)^2}$$

when $|z| > 1$, for $\dfrac{\displaystyle\sum_{k=0}^{n-1} \frac{1}{z^k}}{\displaystyle\sum_{k=0}^{n} \frac{1}{z^k}} \longrightarrow 1$ when $|z| > 1$ and $n \longrightarrow +\infty$.

Hence,

$$\sum_{n=1}^{\infty} \frac{z^{n-1}}{(1-z^n)\,(1-z^{n+1})} = \frac{1}{z(1-z)^2} \quad \text{when } |z| > 1. \quad \text{Q.E.D.}$$

CHAPTER III

SUMMARY

We have seen from Chapter I the condition necessary for uniform convergence is that given $\epsilon > 0$, there exists a positive integer $N = N(\epsilon)$, such that the $|S_n(z) - S(z)| < \epsilon$ for all $n > N$ and z belongs to a region G. Thus we see that uniform convergence is defined on a set whereas convergence is defined at a point.

Now we will investigate what connection the series

$$\sum_{n=1}^{\infty} \frac{z^{n-1}}{(1-z^n)(1-z^{n+1})} \equiv \begin{cases} \dfrac{1}{(1-z)^2}, & \text{when } |z| < 1 \\[2ex] \dfrac{1}{z(1-z)^2}, & \text{when } |z| > 1 \end{cases}$$

have with the theory of uniform convergence.

It has been shown previously that this series converges for all z, such that $|z| < 1$ and $|z| > 1$. But outside of the regions $|z| < 1$ and $|z| > 1$, we have divergence of this series. Thus we see that the series

$$\sum_{n=1}^{\infty} \frac{z^{n-1}}{(1-z^n)(1-z^{n+1})}$$

converges to a definite sum, when z is restricted to $|z| < 1$ and $|z| > 1$.

Since

$$S_n(z) = \frac{1}{(1-z)^2}\left\{\frac{\sum_{k=0}^{n-1} z^k}{\sum_{k=0}^{n} z^k}\right\},$$

convergence of this sequence depends totally on the behavior of the sums $\sum_{k=0}^{n} z^k$ and $\sum_{k=0}^{n-1} z^k$. Since both behave similar, we will consider one.

Observe

$$\lim_{n\to+\infty} \sum_{k=0}^{n} z^k = 1 + z + z^2 + \ldots + z^n + \ldots = \frac{1}{1-z},$$

when $|z| < 1$. This series converges for all z such that $|z| < 1$. Since $\left\{\sum_{k=0}^{n} z^k\right\}$ converges for all z such that the $|z| < 1$,

$$S_n(z) = \frac{1}{(1-z)^2}\left\{\frac{\sum_{k=0}^{n-1} z^k}{\sum_{k=0}^{n} z^k}\right\}$$

converges for all z such that the $|z| < 1$. If z is bound away from 1, then $S_n(z)$ will converge uniformly in this region; i.e., let $\rho < 1$ and arbitrary. Therefore, $S_n(z)$ converges uniformly for all z, such that $|z| \leq \rho$.[1]

Since the sequence $\left\{\sum_{k=0}^{n} z^k\right\}$ converges uniformly for all z such that $|z| \leq \rho$, we have that $S_n(z)$ converges uniformly to $\frac{1}{(1-z)^2}$ when $|z| \leq \rho$, for $\rho < 1$.

Similarly, we have

$$S_n(z) = \frac{1}{z(1-z)^2}\left\{\frac{\sum_{k=0}^{n-1} \frac{1}{z^k}}{\sum_{k=0}^{n} \frac{1}{z^k}}\right\}$$

converges for all z such that the $|z| > 1$. But convergence of this sequence depends wholly on the behavior of the sum $\sum_{k=0}^{n} \frac{1}{z^k}$. Since this sum converges for all z such that the $|z| > 1$, $S_n(z)$ converges for all z such that $|z| > 1$.

If $\rho > 1$ and if $|z| \geq \rho$, then $S_n(z)$ converges uniformly in $|z| \geq \rho$. For the sequence

[1]K. Knopp, Theory of Function, New York, [1945], p. 72.

$\left\{\sum_{k=0}^{n} \frac{1}{z^k}\right\}$ converges uniformly for all z such that the $|z| \geq \rho$ and $\rho > 1$. Hence $S_n(z)$ converges uniformly to $\frac{1}{z(1-z)^2}$ when $|z| \geq \rho$. Q.E.D.

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